Abelian Divisibility Sequences
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Arithmetic of Low-Dimensional Abelian Varieties
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Divisibility Sequences

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Classical examples divisibility sequences include:

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the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, \ldots .
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An **elliptic divisibility sequence** (EDS) is formed from an elliptic curve \(E/\mathbb{Q}\) and a non-torsion point \(P \in E(\mathbb{Q})\) by writing

\[nP = \left( \frac{A_n(P)}{D_n(P)^2}, \frac{B_n(P)}{D_n(P)^3} \right).\]

The sequence \((D_n(P))_{n \geq 1}\) is an EDS.
Divisibility Sequences over Dedekind Domains

More generally, if $R$ is a Dedekind domain, we define an $R$-divisibility sequence to be a sequence of ideals $(\mathfrak{D}_n)_{n \geq 1}$ such that $m | n \implies \mathfrak{D}_m | \mathfrak{D}_n$. 
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In this way we can define an EDS, for example, by factoring the ideal generated by $x(nP)$ in the form

$$x(nP)R = \mathfrak{A}_n(P)\mathfrak{D}_n(P)^{-2}$$

and taking the sequence

$$(\mathfrak{D}_n(P))_{n \geq 1}.$$
Reformulating EDS

Let $E/K$ be an elliptic curve, let $P \in E(K)$, and let $\mathcal{E}/R$ be a Néron model for $E/K$. Then the EDS

$$(\mathcal{D}_n(P))_{n \geq 1}$$

is characterized by noting that for each prime ideal $\mathfrak{p}$ of $R$, we have*

$$\text{ord}_{\mathfrak{p}} \mathcal{D}_n(P) = \left( \text{largest } k \text{ so that } nP \equiv \mathcal{O} \pmod{\mathfrak{p}^k} \right).$$

* Maybe not quite right at primes of bad reduction.
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Or we can simply say that $\mathfrak{D}_n(P)$ is the largest ideal (ordered by divisibility) such that

$$nP \equiv \mathcal{O} \pmod{\mathfrak{D}_n(P)}.$$
Abelian Divisibility Sequences: Type I

In general:

- $R$ a Dedekind domain.
- $K$ the fraction field of $R$.
- $A/K$ an abelian variety.
- $\mathcal{A}/R$ a Néron model for $A/K$.
- $P \in A(K)$ a non-torsion point.

The **abelian divisibility sequence** (ADS) for the pair $(A, P)$ is the sequence of ideals $(\mathfrak{D}_n(P))_{n \geq 1}$ defined by the property that $\mathfrak{D}_n(P)$ is the largest ideal satisfying

$$nP \equiv O \pmod{\mathfrak{D}_n(P)}.$$
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Alternatively, letting $\pi : \mathcal{A} \to \text{Spec}(R)$, we can define $\mathfrak{D}_n(P)$ via arithmetic intersection theory,

$$\mathfrak{D}_n(P) := \pi_*(nP \cdot \mathcal{O}) = (nP)^*(\mathcal{O}) \in \text{Div}(\text{Spec}(R)).$$
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Alternatively, letting $\pi : \mathcal{A} \to \text{Spec}(R)$, we can define $\mathcal{D}_n(P)$ via arithmetic intersection theory,

$$\mathcal{D}_n(P) := \pi_*(nP \cdot O) = (nP)^*(O) \in \text{Div}(\text{Spec}(R)).$$

**Exercise**: Prove that $(\mathcal{D}_n(P))$ is a divisibility sequence.
Growth Rates

A $\mathbb{G}_m$-divisibility sequence $\mathcal{D} = (D_n)$ such as $a^n - b^n$ or the Fibonacci sequence grows exponentially,

$$\lim_{n \to \infty} |D_n|^{1/n} > 1.$$
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Elliptic divisibility sequences $\mathcal{D} = (\mathcal{D}_n(P))$ grow even faster,

$$\lim_{n \to \infty} \left( \frac{N_{K/Q}}{\mathcal{D}_n(P)} \right)^{1/n^2} > 1. \quad (\ast)$$

Two remarks about elliptic divisibility sequences:

- The limit in (\ast) is $\hat{H}_E(P)$, i.e.,

  $$N_{K/Q} \mathcal{D}_n(P) \approx \hat{H}_E(P)^n = \hat{H}_E(nP).$$

- The proof uses a deep, ineffective theorem of Siegel.
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The height of $nP$ on an abelian variety grows at a similar rate, but co-dimension considerations suggest that $\mathcal{D}_n(P)$ might not grow that fast.
Growth Rates of ADS: A Conjecture

**Conjecture 1.** Let $A/K$ be an abelian variety of dimension $\geq 2$, and let $P \in A(K)$ be a point such that $\mathbb{Z}P$ is Zariski dense in $A$. Then

$$\lim_{n \to \infty} \left( N_{K/Q} D_n(P) \right)^{1/n^2} = 1.$$
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The conjecture says that in dimension $\geq 2$, an ADS grows more slowly than the heights of the points in the sequence $nP$. 
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The conjecture says that in dimension $\geq 2$, an ADS grows more slowly than the heights of the points in the sequence $nP$.

**Theorem.** Conjecture 1 follows from Vojta’s conjecture applied to $A$ blown up at $\mathcal{O}$. 
A Multiplicative Analogue to Conjecture 1

Here is a $\mathbb{G}_m$ analogue. We replace $A$ by $\mathbb{G}_m^2$ and $P \in A(K)$ with $(a, b) \in \mathbb{G}_m^2(\mathbb{Q})$. The associated divisibility sequence

$$\gcd(a^n - 1, b^n - 1)$$

measures the “arithmetic distance” from $(a, b)^n$ to $(1, 1)$. 
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**Theorem.** (Bugeaud–Corvaja–Zannier 2003) Let $a, b \in \mathbb{Z}$ with $|a| > |b| > 1$. Then

\[ \lim_{n \to \infty} \gcd(a^n - 1, b^n - 1)^{1/n} = 1. \]

([BCZ] result is more general. See also work of A. Levin.)
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The proof uses Schmidt’s subspace theorem and is surprisingly intricate, even for $a = 3$ and $b = 2$.

**Challenge:** Give an elementary proof that

$$\gcd(3^n - 1, 2^n - 1)^{1/n} \to 1.$$
Growth Rates of ADS: Another Conjecture

Conjecture 1 says that an ADS does not grow too fast. The next conjecture says that for many $n$, it doesn’t grow at all!
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**Conjecture 2.** Let $A/K$ be an abelian variety of dimension $\geq 2$, and let $P \in A(K)$ be a point such that $\mathbb{Z}P$ is Zariski dense in $A$. Then there is a constant $C = C(A/K, P)$ with the property that

\[ (*) \quad |N_{K/\mathbb{Q}} \mathcal{D}_n(P)| \leq C \quad \text{for infinitely many } n \geq 1. \]
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**Bolder Conjecture:** The set of primes $n$ such that the inequality (*) holds is a set of positive density.
Growth Rates of ADS: Another Conjecture

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**Bolder Conjecture:** The set of primes \( n \) such that the inequality \((*)\) holds is a set of positive density.

**Even Bolder Conjecture Question:** The set of primes \( n \) such that the inequality \((*)\) holds is a set of density 1.
A Multiplicative Analogue to Conjecture 2

It is surprising to me that this conjecture wasn’t formulated until quite recently.

**Conjecture.** (Ailon–Rudnick 2004) Let \(a, b \in \mathbb{Z}\) with \(|a| > |b| > 1\). Then there are infinitely many values of \(n \geq 1\) such that

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**Theorem.** (Ailon–Rudnick 2004) Let $a(T), b(T) \in \mathbb{C}[T]$ be multiplicatively independent modulo $\mathbb{C}^*$. Then there is a $c(T) \in \mathbb{C}[t]$ such that

$$\text{gcd}(a(T)^n - 1, b(T)^n - 1) \mid c(T) \quad \text{for all } n \geq 1.$$
Experiments

I’ve gathered a fair amount of data for the $\mathbb{G}_m^2$ conjecture, i.e., Ailon–Rudnick’s conjecture for

$$\gcd(a^n - 1, b^n - 1) = \gcd(a - 1, b - 1),$$

which I will display on the next slide.
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It would be interesting to do some experiments using simple abelian surfaces.
Question: Does the frequency go to 100%?
Fast Growth Versus Slow Growth

As we have seen the growth rate of the ADS associated to an abelian variety $A$ is

- **Fast** if $\dim(A) = 1$;
- **Slow** if $\dim(A) \geq 2$ (conjecturally).
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There are a number of reasons why fast-growing divisibility sequences are useful, including:

- Existence of primitive prime divisors (defined later);
- Applications to logic, Hilbert’s 10th problem;
- Applications to cryptography based on discrete logarithms and/or pairings.
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How can we get fast-growing, geometrically defined, abelian divisibility sequences in high dimension?
Abelian Divisibility Sequences: Type II

Let $\pi : \mathcal{A} \to \text{Spec}(R)$ be a Néron model for $A/K$. For a point $P \in A(K)$ on the generic fiber, let $\overline{P} \subset \mathcal{A}$ be the closure of $P$. We defined the Type I ADS for $(A, P)$ to be the sequence of ideals

$$\mathcal{D}_n(P) := \pi_*(\lbrack n \rbrack P \cdot \mathcal{O}) = \pi_*(\overline{P} \cdot [n]^* \mathcal{O}).$$

This is small when $\dim(A) \geq 2$ because

$$\dim(A) \geq 3, \quad \text{while} \quad \dim \overline{P} = \dim \mathcal{O} = 1.$$
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To obtain a *fast-growing sequence*, we should replace $P$ and/or $\mathcal{O}$ with a higher-dimensional variety.
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To obtain a \textit{fast-growing sequence}, we should replace $P$ and/or $\mathcal{O}$ with a higher-dimensional variety.

And in order to get a \textit{divisibility sequence}, we need to replace $P$, not $\mathcal{O}$.
Abelian Divisibility Sequences: Type II

**Definition:** Let $X \subset A_K$ be an irreducible codimension 1 subvariety defined over $K$, and let $\overline{X}$ be its closure in $A$. The abelian divisibility sequence for the pair $(A, X)$ is the sequence of ideals

$$\mathcal{O}_n(X) := \pi_*(\overline{X} \cdot [n]^*\mathcal{O}).$$

* Need to be a bit careful if $\overline{X}$ contains a component of $[n]^*\mathcal{O}$. 
Abelian Divisibility Sequences: Type II

**Definition:** Let $X \subset A_K$ be an irreducible codimension 1 subvariety defined over $K$, and let $\overline{X}$ be its closure in $A$. The **abelian divisibility sequence** for the pair $(A, X)$ is the sequence of ideals\(^*\)

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\]

\(^*\) Need to be a bit careful if $\overline{X}$ contains a component of $[n]^* \mathcal{O}$.

**Conjecture.** If $A$ is simple, or more generally if $X$ contains no translates of abelian subvarieties, then $\mathcal{D}_n(X)$ is fast-growing:

\[
\liminf_{n \to \infty} \left| N_{K/\mathbb{Q}} \mathcal{D}_n(X) \right|^{1/n^{2\dim A}} > 1.
\]
Tori Divisibility Sequences: Type II

The analogous problem for $\mathbb{G}^N_m$ is solved.

**Theorem.** (Habegger, Dimitrov, 2016) Let $f \in R[T_1^{\pm 1}, \ldots, T_N^{\pm 1}]$ be a Laurent polynomial, and let $X_f \subset \mathbb{G}^N_m$ be the associated divisor. Then

$$\lim_{n \to \infty} \left| N_{K/\mathbb{Q}} \mathfrak{D}_n(X_f) \right|^{1/n^N} = \text{MahlerMeasure}(f).$$
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$$\mathcal{D}_n(X_f) := \prod_{\zeta_1, \ldots, \zeta_N \in \mu_n} f(\zeta_1, \ldots, \zeta_N).$$
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The analogous problem for $\mathbb{G}_m^N$ is solved.

**Theorem.** (Habegger, Dimitrov, 2016) Let $f \in R[T^{\pm 1}_1, \ldots, T^{\pm 1}_N]$ be a Laurent polynomial, and let $X_f \subset \mathbb{G}_m^N$ be the associated divisor. Then

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**Remark.** The theorem is false if we allow $f$ to have $\mathbb{C}$ coefficients. Even for $N = 1$, the theorem requires some sort of estimate coming from linear-forms-in-logarithms, because we need to know that algebraic numbers cannot be too closely approximated by roots of unity.
Primitive Prime Divisors and Zsigmondy Sets

Question: Which terms contain a “new” prime divisor?

Definition: Let $\mathcal{D} := (\mathcal{D}_n)_{n \geq 1}$ be a sequence of ideals. A primitive prime divisor of $\mathcal{D}_n$ is a prime ideal $\mathfrak{p}$ satisfying

$$\mathfrak{p} \mid \mathcal{D}_n \quad \text{and} \quad \mathfrak{p} \nmid \mathcal{D}_m \quad \text{for all} \ m < n.$$
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The **Zsigmondy set of** \( \mathcal{D} \) specifies the terms that do not have a primitive prime divisor:

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\mathcal{Z}(\mathcal{D}) := \{ n \geq 1 : \mathcal{D}_n \text{ has no primitive prime divisors} \}.
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**Some Sample Results:**

- Bang/Zsigmondy (1886/92): $\mathcal{Z}(a^n - b^n) \subseteq \{1, 2, 6\}$.
- Carmichael (1913): $\mathcal{Z}(\text{Fibonacci sequence}) = \{1, 2, 6, 12\}$.
  $$\mathcal{Z}(\text{Lucas/Lehmer sequence}) \subset \{1, 2, \ldots, 30\}.$$
- JS (1988): $\mathcal{Z}(\text{elliptic divisibility sequence})$ is finite.
Primitive Prime Divisors in Abelian Divisibility Sequences

There are two ingredients that go into proofs that the Zsigmondy set of a sequence $\mathcal{D} = (\mathcal{D}_n)_{n \geq 1}$ is finite:

- The sequence grows rapidly in norm, e.g.,
  \[ \log N_{K/\mathbb{Q}} \mathcal{D}_n \gg n^\delta \quad \text{for some } \delta > 1. \]

- The $p$-divisibility does not grow too rapidly, e.g., let $r$ be the smallest index with $p | \mathcal{D}_r$, then
  \[ \text{ord}_p \mathcal{D}_{nr} = \text{ord}_p \mathcal{D}_r + O(\text{ord}_p(n)). \]
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  \[ \text{ord}_p \mathcal{D}_{nr} = \text{ord}_p \mathcal{D}_r + O\left(\text{ord}_p(n)\right). \]

For ADS of Type I, the Ailon–Rudnick conjecture implies that $\mathcal{Z}(\mathcal{D})$ is infinite.
Primitive Prime Divisors in Abelian Divisibility Sequences

There are two ingredients that go into proofs that the Zsigmondy set of a sequence $\mathcal{D} = (\mathcal{D}_n)_{n \geq 1}$ is finite:

- The sequence grows rapidly in norm, e.g.,
  \[ \log N_{K/\mathbb{Q}} \mathcal{D}_n \gg n^{\delta} \quad \text{for some } \delta > 1. \]

- The $p$-divisibility does not grow too rapidly, e.g., let $r$ be the smallest index with $p \mid \mathcal{D}_r$, then
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For ADS of Type I, the Ailon–Rudnick conjecture implies that $\mathcal{Z}(\mathcal{D})$ is infinite.

For ADS of Type II, we conjecturally have rapid norm growth, but $p$-divisibility when $\dim \geq 2$ is much less regular than for $\dim = 1$. Again, I’ve done experiments and have weak partial results for $\mathbb{G}_m^2$, but it would be very interesting to gather data for abelian surfaces.
Primes in Divisibility Sequences

Let $\mathcal{D} = (D_n)_{n \geq 1}$ be a divisibility sequence in $\mathbb{Z}$.

**Natural Question:** Are there infinitely many primes in the sequence $|D_n/D_1|$?
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**Natural Question**: Are there infinitely many primes in the sequence $|D_n/D_1|$?

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**Three Guesses**:

- For a $\mathbb{G}_m$-sequence, which typically satisfies
  \[ \log |D_n| \gg \ll n, \]
  we expect $D_p/D_1$ to be prime for infinitely many $p$.
  *Example*: Mersenne sequence $D_n = 2^n - 1$.

- For an EDS, which typically satisfies
  \[ \log |D_n| \gg \ll n^2, \]
  we expect $D_p/D_1$ to be prime for only finitely many $p$.

- Ditto for higher dimensional Type II ADS with
  \[ \log |D_n| \gg \ll n^d. \]
I want to thank the organizers,
   Jenn, Noam, Brendan,
   Bjorn, Drew, and John,
for inviting me to speak, and to thank you for your attention.
Abelian Divisibility Sequences
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